21. **(a)** $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

$= \lim_{h\to 0}\frac{\frac{1}{a+h-1}-\frac{1}{a-1}}{h}$

$= \lim_{h\to 0}\frac{(a-1)-(a+h-1)}{h(a-1)(a+h-1)}$

$= \lim_{h\to 0}-\frac{1}{(a-1)(a+h-1)}$

$= -\frac{1}{(a-1)^2}$

(b) The slope of the tangent is always negative. The tangents are very steep near $x = 1$ and nearly horizontal as a moves away from the origin.

22. **(a)** $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

$= \lim_{h\to 0}\frac{[9-(a+h)^2]-(9-a^2)}{h}$

$= \lim_{h\to 0}\frac{9-a^2-2ah-h^2-9+a^2}{h}$

$= \lim_{h\to 0}\frac{-2ah-h^2}{h}$

$= \lim_{h\to 0}(-2a-h)$

$= -2a$

(b) The slope of the tangent steadily decreases as a increases.

23. $\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}$

$= \lim_{h\to 0}\frac{3(1+h)-7-(3\cdot 1-7)}{h}$

$= \lim_{h\to 0}\frac{3+3h-7-(-4)}{h}$

$= \lim_{h\to 0}\frac{3h}{h}$

$= \lim_{h\to 0} 3$

$= 3$

The instantaneous rate of change is 3 ft/sec.

24. $\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}$

$= \lim_{h\to 0}\frac{3(3+h)^2+2(3+h)-(3\cdot 3^2+2\cdot 3)}{h}$

$= \lim_{h\to 0}\frac{3(9+6h+h^2)+6+2h-33}{h}$

$= \lim_{h\to 0}\frac{20h+3h^2}{h}$

$= \lim_{h\to 0}(20+3h)$

$= (20+3\cdot 0)$

$= 20$

The instantaneous rate of change is 20 ft/sec.

25. $\lim_{h\to 0}\frac{f(t+h)-f(t)}{h} = \lim_{h\to 0}\frac{\frac{(2+h)+1}{2+h}-\frac{2+1}{2}}{h}$

$= \lim_{h\to 0}\frac{\frac{2(3+h)-3(2+h)}{2(2+h)}}{h}$

$= \lim_{h\to 0}\frac{\frac{-h}{2(2+h)}}{h}$

$= \lim_{h\to 0}\frac{-1}{2(2+h)}$

$= \frac{-1}{2(2+0)}$

$= -\frac{1}{4}$

The instantaneous rate of change is $-\frac{1}{4}$ ft/sec.

26. $\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}$

$= \lim_{h\to 0}\frac{(2+h)^3-1-(2^3-1)}{h}$

$= \lim_{h\to 0}\frac{8+12h+6h^2+h^3-1-7}{h}$

$= \lim_{h\to 0}\frac{12h+6h^2+h^3}{h}$

$= \lim_{h\to 0}(12+6h+h^2)$

$= 12+6\cdot 0+0^2$

$= 12$

The instantaneous rate of change is 12 ft/sec.

27. Let $f(t) = 100 - 4.9t^2$.

$$\lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h\to 0} \frac{[100-4.9(2+h)^2]-[100-4.9(2)^2]}{h}$$
$$= \lim_{h\to 0} \frac{100-19.6-19.6h-4.9h^2-100+19.6}{h}$$
$$= \lim_{h\to 0} (-19.6-4.9h)$$
$$= -19.6$$

The object is falling at a speed of 19.6 m/sec.

28. Let $f(t) = 3t^2$.

$$\lim_{h\to 0} \frac{f(10+h)-f(10)}{h}$$
$$= \lim_{h\to 0} \frac{3(10+h)^2-300}{h}$$
$$= \lim_{h\to 0} \frac{300+60h+3h^2-300}{h}$$
$$= \lim_{h\to 0} (60+3h)$$
$$= 60$$

The rocket's speed is 60 ft/sec.

29. Let $f(r) = \pi r^2$, the area of a circle of radius r.

$$\lim_{h\to 0} \frac{f(3+h)-f(3)}{h}$$
$$= \lim_{h\to 0} \frac{\pi(3+h)^2-\pi(3)^2}{h}$$
$$= \lim_{h\to 0} \frac{9\pi+6\pi h+\pi h^2-9\pi}{h}$$
$$= \lim_{h\to 0} (6\pi+\pi h)$$
$$= 6\pi$$

The area is changing at a rate of 6π in^2/in., that is, 6π square inches of area per inch of radius.

30. Let $f(r) = \frac{4}{3}\pi r^3$.

$$\lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h\to 0} \frac{\frac{4}{3}\pi(2+h)^3-\frac{4}{3}\pi(2)^3}{h}$$
$$= \frac{4}{3}\pi \lim_{h\to 0} \frac{(2+h)^3-2^3}{h}$$
$$= \frac{4}{3}\pi \lim_{h\to 0} \frac{8+12h+6h^2+h^3-8}{h}$$
$$= \frac{4}{3}\pi \lim_{h\to 0} (12+6h+h^2)$$
$$= \frac{4}{3}\pi \cdot 12$$
$$= 16\pi$$

The volume is changing at a rate of 16π in^3/in., that is, 16π cubic inches of volume per inch of radius.

31. $$\lim_{h\to 0} \frac{s(1+h)-s(1)}{h}$$
$$= \lim_{h\to 0} \frac{1.86(1+h)^2-1.86(1)^2}{h}$$
$$= \lim_{h\to 0} \frac{1.86+3.72h+1.86h^2-1.86}{h}$$
$$= \lim_{h\to 0} (3.72+1.86h)$$
$$= 3.72$$

The speed of the rock is 3.72 m/sec.

32. $$\lim_{h\to 0} \frac{s(2+h)-s(2)}{h}$$
$$= \lim_{h\to 0} \frac{11.44(2+h)^2-11.44(2)^2}{h}$$
$$= \lim_{h\to 0} \frac{45.76+45.76h+11.44h^2-45.76}{h}$$
$$= \lim_{h\to 0} (45.76+11.44h)$$
$$= 45.76$$

The speed of the rock is 45.76 m/sec.

33. First, find the slope of the tangent at $x = a$.

$$\lim_{h\to0}\frac{f(a+h)-f(a)}{h}$$
$$=\lim_{h\to0}\frac{[(a+h)^2+4(a+h)-1]-(a^2+4a-1)}{h}$$
$$=\lim_{h\to0}\frac{a^2+2ah+h^2+4a+4h-1-a^2-4a+1}{h}$$
$$=\lim_{h\to0}\frac{2ah+h^2+4h}{h}$$
$$=\lim_{h\to0}(2a+h+4)$$
$$=2a+4$$

The tangent at $x = a$ is horizontal when $2a + 4 = 0$, or $a = -2$. The tangent line is horizontal at $(-2,\ f(-2)) = (-2, -5)$.

34. First, find the slope of the tangent at $x = a$.

$$\lim_{h\to0}\frac{f(a+h)-f(a)}{h}$$
$$=\lim_{h\to0}\frac{[3-4(a+h)-(a+h)^2]-(3-4a-a^2)}{h}$$
$$=\lim_{h\to0}\frac{3-4a-4h-a^2-2ah-h^2-3+4a+a^2}{h}$$
$$=\lim_{h\to0}\frac{-4h-2ah-h^2}{h}$$
$$=\lim_{h\to0}(-4-2a-h)$$
$$=-4-2a$$

The tangent at $x = a$ is horizontal when $-4 - 2a = 0$, or $a = -2$. The tangent line is horizontal at $(-2, f(-2)) = (-2, 7)$.

35. **(a)** From Exercise 21, the slope of the curve at $x = a$, is $-\frac{1}{(a-1)^2}$. The tangent has slope -1 when $-\frac{1}{(a-1)^2}=-1,$ which gives $(a-1)^2 = 1$, so $a = 0$ or $a = 2$. Note that $y(0)=\frac{1}{0-1}=-1$ and $y(2)=\frac{1}{2-1}=1,$ so we need to find the equations of lines of slope -1 passing through $(0, -1)$ and $(2, 1)$, respectively.

At $x=0$: $y=-1(x-0)-1$

$y=-x-1$

At $x=2$: $y=-1(x-2)+1$

$y=-x+3$

(b) The normal has slope $-\frac{1}{-1}=1$ since the tangent has slope -1, so we again need to find lines through $(0, -1)$ and $(2, 1)$, this time using slope 1.

At $x=0: y=1(x-0)-1$

$y=x-1$

At $x=2: y=1(x-2)+1$

$y=x-1$

There is only one such line. It is normal to the curve at two points and its equation is $y = x - 1$.

36. Consider a line that passes through $(1, 12)$ and a point $(a, 9 - a^2)$ on the curve. Using the result of Exercise 22, this line will be tangent to the curve at a if its slope is $-2a$.

$$\frac{(9-a^2)-12}{a-1}=-2a$$
$$9-a^2-12=-2a(a-1)$$
$$-a^2-3=-2a^2+2a$$
$$a^2-2a-3=0$$
$$(a+1)\ (a-3)=0$$
$$a=-1 \text{ or } a=3$$

At $a=-1$ (or $x=-1$), the slope is $-2(-1)=2.$

$y=2(x-1)+12$

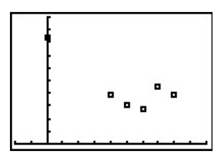
$y=2x+10$

At $a=3$ (or $x=3$), the slope is $-2(3)=-6.$

$y=-6(x-1)+12$

$y=-6x+18$

37. **(a)**



[–2, 10] by [0, 1000]

(b) slope of $PQ_1=\frac{389-381}{8-4}=\frac{8}{4}=2$

slope of $PQ_2=\frac{389-313}{8-5}=\frac{76}{3}=25.33$

slope of $PQ_3=\frac{389-448}{8-7}=\frac{-59}{1}=-59$

38. **(a)**
$$\frac{\$(690.3-404.8)\text{ billion}}{(2009-2003)\text{ years}}$$
$$=\frac{\$285.5\text{ billion}}{6\text{ years}}$$
$$=\$45.583\ \frac{\text{billion}}{\text{year}}$$

(b) $\dfrac{\$(616.1-495.3)\text{ billion}}{(2008-2005)\text{ years}}$

$= \$40.267\ \dfrac{\text{billion}}{\text{year}}$

(c) $\dfrac{\$(690.3-616.1)\text{ billion}}{(2009-2008)\text{ years}} = \$74.2\ \dfrac{\text{billion}}{\text{year}}$

(d) One possible reason is that the war in Afghanistan and increased spending to prevent terrorist attacks in the U.S. caused an unusual increase in defense spending.

39. True. The normal line is perpendicular to the tangent line at the point.

40. False. There's no tangent at $x = 0$ because f has no slope at $x = 0$.

41. D; $\dfrac{-3-5}{-1-2} = \dfrac{8}{3}$

42. E; $\dfrac{f(3)-f(1)}{3-1} = \dfrac{3^2+3-1^2-1}{2} = \dfrac{10}{2} = 5$

43. C;

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\left(\frac{2}{x+h}-\frac{2}{x}\right)\frac{1}{h}$$
$$= \lim_{h\to 0}\frac{2x-2x-2h}{x^2+hx}\left(\frac{1}{h}\right)$$
$$= -\frac{2}{x^2}$$

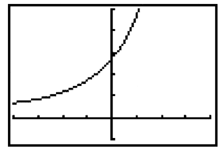
$$y = -\frac{2}{(1)^2} = -2$$
$$y = m(x-x_1)+y_1$$
$$y = -2(x-1)+2$$
$$y = -2x+4$$

44. A; From 39, $m_2 = -\dfrac{1}{m_1}$

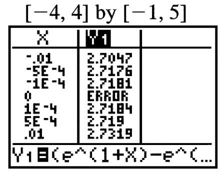
$$m_2 = \frac{1}{2}$$
$$y = \frac{1}{2}(x-1)+2$$
$$y = \frac{1}{2}x+\frac{3}{2}$$

45. **(a)** $\dfrac{f(1+h)-f(1)}{h} = \dfrac{e^{1+h}-e}{h}$

(b)



[−4, 4] by [−1, 5]



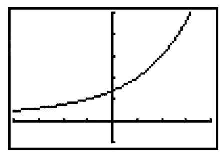
Limit ≈ 2.718

(c) They're about the same.

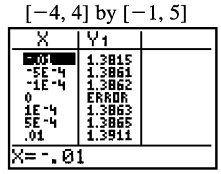
(d) Yes, it has a tangent whose slope is about e.

46. **(a)** $\dfrac{f(1+h)-f(1)}{h} = \dfrac{2^{1+h}-2}{h}$

(b)



[−4, 4] by [−1, 5]

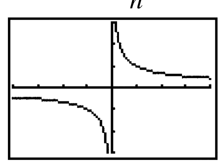


Limit ≈ 1.386

(c) They're about the same.

(d) Yes, it has a tangent whose slope is about ln 4.

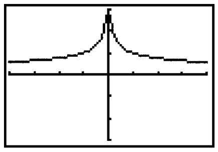
47. Let $f(x) = x^{2/5}$. The graph of $y = \dfrac{f(0+h)-f(0)}{h} = \dfrac{f(h)}{h}$ is shown.



[−4, 4] by [−3, 3]

The left- and right-hand limits are $-\infty$ and ∞, respectively. Since they are not the same, the curve does not have a vertical tangent at $x = 0$. No.

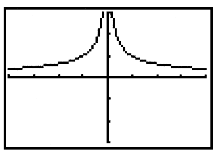
48. Let $f(x) = x^{3/5}$. The graph of $y = \frac{f(0+h)-f(0)}{h} = \frac{f(h)}{h}$ is shown.



[−4, 4] by [−3, 3]

Yes, the curve has a vertical tangent at $x = 0$ because $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \infty$.

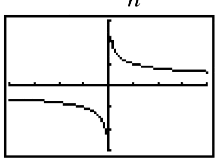
49. Let $f(x) = x^{1/3}$. The graph of $y = \frac{f(0+h)-f(0)}{h} = \frac{f(h)}{h}$ is shown.



[−4, 4] by [−3, 3]

Yes, the curve has a vertical tangent at $x = 0$ because $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \infty$.

50. Let $f(x) = x^{2/3}$. The graph of $y = \frac{f(0+h)-f(0)}{h} = \frac{f(h)}{h}$ is shown.



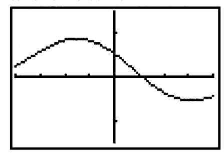
[−4, 4] by [−3, 3]

The left- and right-hand limits are $-\infty$ and ∞, respectively. Since they are not the same, the curve does not have a vertical tangent at $x = 0$. No.

51. This function has a tangent with slope zero at the origin.
It is sandwiched between two functions, $y = x^2$ and $y = -x^2$, both of which have slope zero at the origin.
Looking at the difference quotient, $-h \le \frac{f(0+h)-f(0)}{h} \le h$, so the Sandwich Theorem tells us the limit is 0.

52. This function does not have a tangent line at the origin. As the function oscillates between $y = x$ and $y = -x$ infinitely often near the origin, there are an infinite number of difference quotients (secant line slopes) with a value of 1 and with a value of -1. Thus the limit of the difference quotient doesn't exist.
The difference quotient is $\frac{f(0+h)-f(0)}{h} = \sin\frac{1}{h}$ which oscillates between 1 and -1 infinitely often near zero.

53. Let $f(x) = \sin x$. The difference quotient is $\frac{f(1+h)-f(1)}{h} = \frac{\sin(1+h)-\sin(1)}{h}$.
A graph and table for the difference quotient are shown.



[−4, 4] by [−1.5, 1.5]

X	Y1
-.005	.5424
-.002	.54114
-.001	.54072
0	ERROR
.001	.53988
.002	.53946
.005	.5382

Y1■(sin(1+X)−si…

Since the limit as $h \to 0$ is about 0.540, the slope of $y = \sin x$ at $x = 1$ is about 0.540.

54. The average rate of change of f over the interval $[3, 3 + h]$ is $\frac{f(3+h)-f(3)}{(3+h)-3} = \frac{f(3+h)-f(3)}{h}$ which is the difference quotient with $a = 3$.

55. **(a)** If $x = a + h$, then $h = x - a$. Replace $a + h$ with x and h with $x - a$.

$$\frac{f(a+h)-f(a)}{h} = \frac{f(x)-f(a)}{x-a}$$

(b) Depending on the situation, one form might be more convenient to use than the other.

Quick Quiz Sections 2.3 and 2.4

1. D $\frac{f(3)-f(0)}{3-0} = \frac{\sqrt{3+1}-\sqrt{0+1}}{3} = \frac{2-1}{3} = \frac{1}{3}$

2. E; $f(4-h) \approx \frac{3}{4}(4) \approx 3$ where $h \to 0$

$f(4) = 2$

$f(4+h) \approx -4+7 \approx 3$

3. B; $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

$\lim_{h\to 0}\frac{9-(x+h)^2-(9-x^2)}{h}$

$\lim_{h\to 0}\frac{-2xh-h^2}{h}=\lim_{h\to 0}(-2x-h)=-2x$

$y=9-x^2=9-(2)^2$

$y=5$

$y'=-2(2)=-4$

$y=-4(x-2)+5$

$y=-4x+13$

4. **(a)** $f(3)=2(3)-(3)^2=6-9=-3$

(b) $f(3+h)=2(3+h)-(3+h)^2$
$=6+2h-(9+6h+h^2)$
$=-3-4h-h^2$

(c) $\frac{f(3+h)-f(3)}{h}=\frac{-3-4h-h^2-(-3)}{h}$
$=-4-h$

(d) $\lim_{h\to 0}(-4-h)=-4$

Chapter 2 Review Exercises (pp. 96–97)

1. $\lim_{x\to -2}(x^3-2x^2+1)=(-2)^3-2(-2)^2+1=-15$

2. $\lim_{x\to -2}\frac{x^2+1}{3x^2-2x+5}=\frac{(-2)^2+1}{3(-2)^2-2(-2)+5}=\frac{5}{21}$

3. No limit, because the expression $\sqrt{1-2x}$ is undefined for values of x near 4.

4. No limit, because the expression $\sqrt[4]{9-x^2}$ is undefined for values of x near 5.

5. $\lim_{x\to 0}\frac{\frac{1}{2+x}-\frac{1}{2}}{x}=\lim_{x\to 0}\frac{2-(2+x)}{2x(2+x)}$

$=\lim_{x\to 0}\frac{-x}{2x(2+x)}$

$=\lim_{x\to 0}\left(-\frac{1}{2(2+x)}\right)$

$=-\frac{1}{2(2+0)}$

$=-\frac{1}{4}$

6. $\lim_{x\to\pm\infty}\frac{2x^2+3}{5x^2+7}=\lim_{x\to\pm\infty}\frac{2x^2}{5x^2}=\frac{2}{5}$

7. An end behavior model for

$\frac{x^4+x^3}{12x^3+128}$ is $\frac{x^4}{12x^3}=\frac{1}{12}x.$

Therefore

$\lim_{x\to\infty}\frac{x^4+x^3}{12x^3+128}=\lim_{x\to\infty}\frac{1}{12}x=\infty$

$\lim_{x\to -\infty}\frac{x^4+x^3}{12x^3+128}=\lim_{x\to -\infty}\frac{1}{12}x=-\infty$

8. $\lim_{x\to 0}\frac{\sin 2x}{4x}=\frac{1}{2}\lim_{x\to 0}\frac{\sin 2x}{2x}=\frac{1}{2}(1)=\frac{1}{2}$

9. Multiply the numerator and denominator by $\sin x$.

$\lim_{x\to 0}\frac{x\csc x+1}{x\csc x}=\lim_{x\to 0}\frac{x+\sin x}{x}$

$=\lim_{x\to 0}\left(1+\frac{\sin x}{x}\right)$

$=\left(\lim_{x\to 0}1\right)+\left(\lim_{x\to 0}\frac{\sin x}{x}\right)$

$=1+1$

$=2$

10. $\lim_{x\to 0}e^x\sin x=e^0\sin 0=1\cdot 0=0$

11. Let $x=\frac{7}{2}+h,$ where h is in $\left(0,\ \frac{1}{2}\right)$. Then

$\text{int}(2x-1)=\text{int}\left[2\left(\frac{7}{2}\right)+2h-1\right]=\text{int}\ (6+2h)=6,$

because $6+2h$ is in (6, 7). Therefore,
$\lim_{x\to 7/2^+}\text{int}(2x-1)=\lim_{x\to 7/2^+}6=6.$

12. Let $x=\frac{7}{2}+h,$ where h is in $\left(-\frac{1}{2},\ 0\right)$. Then

$\text{int}(2x-1)=\text{int}\left[2\left(\frac{7}{2}\right)+2h-1\right]=\text{int}\ (6+2h)=5,$

because $6+2h$ is in (5, 6). Therefore,
$\lim_{x\to 7/2^-}\text{int}(2x-1)=\lim_{x\to 7/2^-}5=5$

13. Since $\lim_{x\to\infty}(-e^{-x})=\lim_{x\to\infty}e^{-x}=0,$ and $-e^{-x}\le e^{-x}\cos x\le e^{-x}$ for all x, the Sandwich Theorem gives $\lim_{x\to\infty}e^{-x}\cos x=0.$

14. Since the expression x is an end behavior model for both $x+\sin x$ and $x+\cos x$, $\lim_{x\to\infty}\frac{x+\sin x}{x+\cos x}=\lim_{x\to\infty}\frac{x}{x}=1$.

15. Limit exists.

16. Limit exists.

17. Limit exists.

18. Limit does not exist.

19. Limit exists.

20. Limit exists.

21. Yes

22. No

23. No

24. Yes

25. **(a)** $\lim_{x\to 3^-} g(x)=1$

(b) $g(3) = 1.5$

(c) No, since $\lim_{x\to 3^-} g(x)\neq g(3)$.

(d) g is discontinuous at $x = 3$ (and at points not in the domain).

(e) Yes, the discontinuity at $x = 3$ can be removed by assigning the value 1 to $g(3)$.

26. **(a)** $\lim_{x\to 1^-} k(x)=1.5$

(b) $\lim_{x\to 1^+} k(x)=0$

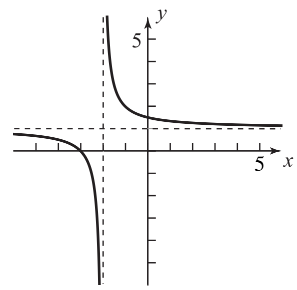
(c) $k(1) = 0$

(d) No, since $\lim_{x\to 1^-} k(x)\neq k(1)$.

(e) k is discontinuous at $x = 1$ (and at points not in the domain).

(f) No, the discontinuity at $x = 1$ is not removable because the one-sided limits are different.

27.

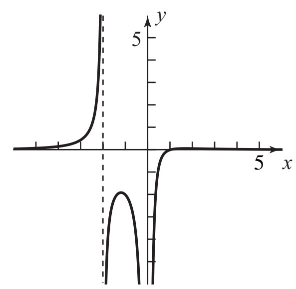


(a) Vertical asymptote: $x = -2$

(b) Left-hand limit $= \lim_{x\to -2^-}\frac{x+3}{x+2}=-\infty$

Right-hand limit: $\lim_{x\to -2^+}\frac{x+3}{x+2}=\infty$

28.



(a) Vertical asymptotes: $x = 0$, $x = -2$

(b) At $x = 0$:

Left-hand limit $= \lim_{x\to 0^-}\frac{x-1}{x^2(x+2)}=-\infty$

Right-hand limit $= \lim_{x\to 0^+}\frac{x-1}{x^2(x+2)}=-\infty$

At $x = -2$:

Left-hand limit $= \lim_{x\to -2^-}\frac{x-1}{x^2(x+2)}=\infty$

Right-hand limit $= \lim_{x\to -2^+}\frac{x-1}{x^2(x+2)}=-\infty$

29. **(a)** At $x = -1$:

Left-hand limit $= \lim_{x\to -1^-} f(x) = \lim_{x\to -1^-}(1) = 1$

Right-hand limit $= \lim_{x\to -1^+} f(x) = \lim_{x\to -1^+}(-x) = 1$

At $x = 0$:

Left-hand limit $= \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-x) = 0$

Right-hand limit $= \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (-x) = 0$

At $x = 1$:

Left-hand limit $= \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (-x) = -1$

Right-hand limit $= \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (1) = 1$

(b) At $x = -1$: Yes, the limit is 1.
At $x = 0$: Yes, the limit is 0.
At $x = 1$: No, the limit doesn't exist because the two one-sided limits are different.

(c) At $x = -1$: Continuous because $f(-1)$ = the limit.
At $x = 0$: Discontinuous because $f(0) \neq$ the limit.
At $x = 1$: Discontinuous because the limit does not exist.

30. **(a)** Left-hand limit $= \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} |x^3 - 4x| = |(1)^3 - 4(1)| = |-3| = 3$

Right-hand limit $= \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2 - 2x - 2) = (1)^2 - 2(1) - 2 = -3$

(b) No, because the two one-sided limits are different.

(c) Every place except for $x = 1$

(d) At $x = 1$

31. Since $f(x)$ is a quotient of polynomials, it is continuous and its points of discontinuity are the points where it is undefined, namely $x = -2$ and $x = 2$.

32. There are no points of discontinuity, since $g(x)$ is continuous and defined for all real numbers.

33. **(a)** End behavior model: $\frac{2x}{x^2}$, or $\frac{2}{x}$

(b) Horizontal asymptote: $y = 0$ (the x-axis)

34. **(a)** End behavior model: $\frac{2x^2}{x^2}$, or 2

(b) Horizontal asymptote: $y = 2$

35. **(a)** End behavior model: $\frac{x^3}{x}$, or x^2

(b) Since the end behavior model is quadratic, there are no horizontal asymptotes.

36. **(a)** End behavior model: $\frac{x^4}{x^3}$, or x

(b) Since the end behavior model represents a nonhorizontal line, there are no horizontal asymptotes.

37. **(a)** Since $\lim_{x\to\infty} \frac{x+e^x}{e^x} = \lim_{x\to\infty}\left(\frac{x}{e^x}+1\right) = 1$, a right end behavior model is e^x.

(b) Since $\lim_{x\to-\infty} \frac{x+e^x}{x} = \lim_{x\to-\infty}\left(1+\frac{e^x}{x}\right) = 1$, a left end behavior model is x.

38. **(a, b)** Note that

$$\lim_{x\to\pm\infty}\left(-\frac{1}{\ln|x|}\right) = \lim_{x\to\pm\infty}\left(\frac{1}{\ln|x|}\right) = 0 \text{ and}$$

$$-\frac{1}{\ln|x|} < \frac{\sin x}{\ln|x|} < \frac{1}{\ln|x|} \quad \text{whenever } |x| > 1.$$

Therefore, the Sandwich Theorem gives $\lim_{x\to\pm\infty} \frac{\sin x}{\ln|x|} = 0$. Hence

$$\lim_{x\to\pm\infty}\frac{\ln|x|+\sin x}{\ln|x|}=\lim_{x\to\pm\infty}\left(1+\frac{\sin x}{\ln|x|}\right)$$
$$=1+0$$
$$=1,$$

so $\ln|x|$ is both a right end behavior model and a left end behavior model.

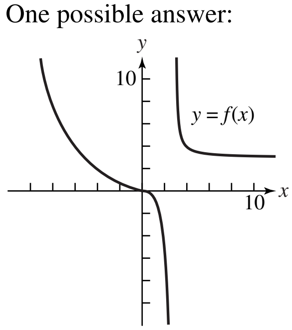
39. $\lim_{x\to3} f(x)=\lim_{x\to3}\frac{x^2+2x-15}{x-3}$
$=\lim_{x\to3}\frac{(x-3)(x+5)}{x-3}$
$=\lim_{x\to3}(x+5)$
$=3+5$
$=8$

Assign the value $k = 8$.

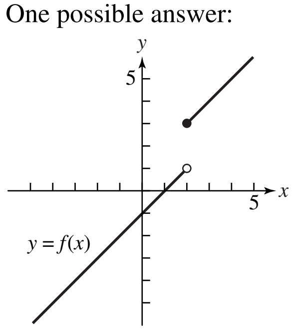
40. $\lim_{x\to0} f(x)=\lim_{x\to0}\frac{\sin x}{2x}=\frac{1}{2}\lim_{x\to0}\frac{\sin x}{x}=\frac{1}{2}(1)=\frac{1}{2}$

Assign the value $k=\frac{1}{2}$.

41. One possible answer:



42. One possible answer:



43. $\frac{f\left(\frac{\pi}{2}\right)-f(0)}{\left(\frac{\pi}{2}\right)-0}=\frac{2-1}{\frac{\pi}{2}}=\frac{2}{\pi}$

44. $\lim_{h\to0}\frac{V(a+h)-V(a)}{h}$
$=\lim_{h\to0}\frac{\frac{1}{3}\pi(a+h)^2H-\frac{1}{3}\pi a^2H}{h}$
$=\frac{1}{3}\pi H\lim_{h\to0}\frac{a^2+2ah+h^2-a^2}{h}$
$=\frac{1}{3}\pi H\lim_{h\to0}(2a+h)$
$=\frac{1}{3}\pi H(2a)$
$=\frac{2}{3}\pi aH$

45. $\lim_{h\to0}\frac{S(a+h)-S(a)}{h}$
$=\lim_{h\to0}\frac{6(a+h)^2-6a^2}{h}$
$=\lim_{h\to0}\frac{6a^2+12ah+6h^2-6a^2}{h}$
$=\lim_{h\to0}(12a+6h)$
$=12a$

46. $\lim_{h\to0}\frac{y(a+h)-y(a)}{h}$
$=\lim_{h\to0}\frac{[(a+h)^2-(a+h)-2]-(a^2-a-2)}{h}$
$=\lim_{h\to0}\frac{a^2+2ah+h^2-a-h-2-a^2+a+2}{h}$
$=\lim_{h\to0}\frac{2ah+h^2-h}{h}$
$=\lim_{h\to0}(2a+h-1)$
$=2a-1$

47. **(a)** $\lim_{h\to0}\frac{f(1+h)-f(1)}{h}$
$=\lim_{h\to0}\frac{[(1+h)^2-3(1+h)]-(-2)}{h}$
$=\lim_{h\to0}\frac{1+2h+h^2-3-3h+2}{h}$
$=\lim_{h\to0}(-1+h)$
$=-1$

(b) The tangent at P has slope -1 and passes through $(1, -2)$.
$y=-1(x-1)-2$
$y=-x-1$

(c) The normal at P has slope 1 and passes through (1, –2).
$y = 1(x-1)-2$
$y = x-3$

48. At $x = a$, the slope of the curve is

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$
$$=\lim_{h\to 0}\frac{[(a+h)^2-3(a+h)]-(a^2-3a)}{h}$$
$$=\lim_{h\to 0}\frac{a^2+2ah+h^2-3a-3h-a^2+3a}{h}$$
$$=\lim_{h\to 0}\frac{2ah-3h+h^2}{h}$$
$$=\lim_{h\to 0}(2a-3+h)$$
$$=2a-3$$

The tangent is horizontal when $2a - 3 = 0$, at $a=\frac{3}{2}\left(\text{or } x=\frac{3}{2}\right)$. Since $f\left(\frac{3}{2}\right)=-\frac{9}{4}$, the point where this occurs is $\left(\frac{3}{2}, -\frac{9}{4}\right)$.

49. **(a)** $p(0)=\frac{200}{1+7e^{-0.1(0)}}=\frac{200}{8}=25$

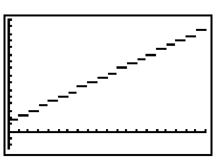
Perhaps this is the number of bears placed in the reserve when it was established.

(b) $\lim_{t\to\infty} p(t)=\lim_{t\to\infty}\frac{200}{1+7e^{-0.1t}}=\frac{200}{1}=200$

(c) Perhaps this is the maximum number of bears which the reserve can support due to limitations of food, space, or other resources. Or, perhaps the number is capped at 200 and excess bears are moved to other locations.

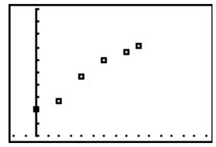
50. **(a)** $f(x)=\begin{cases}3.20-1.35\,\text{int}(-x+1), & 0<x\le 20\\ 0, & x=0\end{cases}$

(Note that we cannot use the formula $f(x) = 3.20 + 1.35 \text{ int } x$, because it gives incorrect results when x is an integer.)

(b) 

[0, 20] by [–5, 32]

f is discontinuous at integer values of x: 0, 1, 2, . . . , 19.

51. **(a)** 

[–2, 15] by [15,000, 20,000]

(b) $PQ_1=\frac{18{,}538-16{,}341}{9-2}=\frac{2197}{7}=313.857$

$PQ_2=\frac{18{,}538-17{,}314}{9-4}=\frac{1224}{5}=244.8$

$PQ_3=\frac{18{,}538-18{,}328}{9-8}=\frac{210}{1}=210$

(c) The slopes in part (b) can be interpreted as the average rates of change. The average rate of change in the population from Q_1 to P, or from 2002 to 2009, is 313.857 thousand people per year. The average rate of change in the population from Q_2 to P, or from 2004 to 2009, is 244.8 thousand people per year. The average rate of change in the population from Q_3 to P, or from 2008 to 2009, is 210 thousand people per year.

(d) Answers will vary. We could argue that the average rate of change in the population from 2008 to 2009, 210 thousand people per year, is a good estimate for the instantaneous rate of change on July 1, 2009, since the data were collected on July 1 each year.

(e) Answers will vary. A linear regression equation for the data is $y = 296x + 16{,}000$, where $x = 0$ represents 2000. Using this model, the population of Florida in 2020 will be $16{,}000 + 296(20) = 21{,}920$ thousand people.

52. Let $A=\lim_{x\to c} f(x)$ and $B=\lim_{x\to c} g(x)$. Then $A + B = 2$ and $A - B = 1$. Adding, we have $2A = 3$, so $A=\frac{3}{2}$, whence $\frac{3}{2}+B=2$, which gives $B=\frac{1}{2}$. Therefore, $\lim_{x\to c} f(x)=\frac{3}{2}$ and $\lim_{x\to c} g(x)=\frac{1}{2}$.

53. **(a)** $x^2-9\ne 0$

All x not equal to –3 or 3.

(b) $x = -3, x = 3$

(c) $\lim\limits_{x\to\infty} = \dfrac{x}{|x^2-9|} = 0$

$y = 0$

(d) The function is odd: $f(-x) = -f(x)$

(e) f is discontinuous at -3 and 3. These are nonremovable discontinuities.

54. **(a)** $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} (x^2 - a^2x)$

$= (2)^2 - a^2(2)$

$= 4 - 2a^2$

(b) $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (4 - 2x^2)$

$= 4 - 2(2)^2$

$= 4 - 8$

$= -4$

(c) For $x \neq 2$, f is continuous. For $x = 2$, we have

$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x) = f(2) = -4$ as long as $a = \pm 2$.

55. **(a)** By inspection, $x^3 - 2x^2 + 1 = 0$ when $x = 1$. Use synthetic division to write $x^3 - 2x^2 + 1 = (x-1)(x^2 - x - 1)$; then use the quadratic formula to find the zeros of $x^2 - x - 1$ to be $\dfrac{1 \pm \sqrt{5}}{2}$. The zeros of f are 1, $\dfrac{1+\sqrt{5}}{2}$, and $\dfrac{1-\sqrt{5}}{2}$.

(b) A right end-behavior model for f is

$y = \dfrac{x^3}{x^2} = x.$

(c) $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{x^3 - 2x^2 + 1}{x^2 + 3} = +\infty$ and

$\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \dfrac{x^3 - 2x^2 + 1}{x^3 + 3x} = 1.$